Graph Isomorphism Problem: Diophantine Algebraic Matrix Riccati, Sylvester Equations

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ABSTRACT

Graph isomorphism problem has been attempted by various researchers. One of the earlier researchers related the problem to solution of structured Diophantine matrix Riccati equation as well as structured Sylvester equation [17]. He provided several necessary conditions for isomorphism of graphs and also a necessary and sufficient condition. We progress those promising approaches and provide a polynomial time algorithm to verify one of the strong necessary conditions. Thus, the algorithms provide correct decision when the input graphs are non-isomorphic. The hope is that connections to algebraic matrix Riccati equations (and their well known solution methods) could provide new approaches to solve the associated matrix Diophantine equations (including the { 0, 1 } matrix solutions). Also, results related to Sylvester equation and matrix Non-Symmetric Algebraic Riccati equation,NARE (including the case where the unknown matrix is orthogonal) are derived.

1. INTRODUCTION:

Graphs naturally arise in many areas of human endeavor such as science, engineering, economics etc. Thus, several interesting problems related to graphs such as minimum cut computation were investigated and efficient algorithms (such as max flow-min cut Theorem based Ford-Fulkerson algorithm) were designed. Several important research problems related to graphs (such as the determination of Hamiltoniam path in a graph) were formulated and efforts were made to solve them (and design computationally efficient algorithms). One such problem is the so called "graph isomorphism problem". There were efforts to solve such a problem i.e. determine if two graphs are isomorphic by a computationally efficient algorithm. Graph isomorphism problem naturally arises in many applications in science and engineering. Thus, efficient polynomial time algorithms for solution of such problem will have significant impact.

This research paper is organized as follows. In Section 2, relevant research literature is reviewed. In Section 3, discovery of matrix equations to solve such a problem are identified. Particularly, it is reasoned that the "structured" algebraic matrix Ricatti equation as well as "structured" Sylvester equation naturally arise in determining isomorphism of two graphs. The research paper concludes in Section 4.

2. Review of Relevant Research Literature:

Early research efforts on the graph isomorphism problem were focused on structured/constrained graphs such as triconnected planar graphs [5]. Also, linear time algorithm for isomorphism of planar graphs was designed [6]. Based on quadratic nonnegative matrix factorization, the problem of graph isomorphism was investigated in [14]. Also, based on algebraic concepts such as permutation groups, graph isomorphism problem was investigated in [11], [12], [13]. One of the most recent contributions to the problem was made by L. Babai [1]. He claimed that a quasi-polynomial time algorithm for determining isomorphism of two graphs [2]. Specifically, Babai showed that graph isomorphism problem can be solved in $\exp(\log n)^{(o(1))}$ time [2] (where n is the number of vertices).

3. Isomorphic Graphs/ Sub-graphs: Matrix Algebraic Riccati and Sylvester Equations:

In the decision problem of determining isomorphism of two graphs using an algorithm, we are led to the following cases:

(i) Graphs are isomorphic and the algorithm declares them isomorphic i.e the algorithmic decision is correct.

(ii) Graphs are isomorphic and the algorithm declares them non-isomorphic i.e the algorithmic decision is wrong

(iii) Graphs are not isomorphic and the algorithm declares them isomorphic i.e. the algorithmic decision is wrong

(iv) Graphs are not isomorphic and the algorithm declares them non-isomorphic i.e the algorithmic decision is correct.

Suppose A, B are the adjacency matrices of two graphs. It readily follows that if they are isomorphic graphs, then

$$B = P\,A\,P^T,\ldots\ldots(1)$$

where P is a permutation matrix. Thus, the isomorphism problem boils down to determining the existence of a permutation matrix P (given the symmetric adjacency matrices A, B) such that the equation (1) is satisfied. The author of [17] derived three necessary conditions for isomorphism of undirected graphs. He also derived a necessary and sufficient condition for isomorphism of two graphs with adjacency matrices { A, B } (in Lemma [3] of the research paper [17]). For the sake of completeness, Lemma [3] in [17] is stated below: In [17], the following notational convention is utilized.

Notation: Consider undirected graphs with adjacency matrices A, B. If the graphs are isomorphic, the matrices A, B are said to be isomorphic.

Lemma 1: The matrices A and B are isomorphic if and only if the the following matrix equation

$$P\,A - B\,P \equiv \bar{0}\ldots\ldots(2)$$

has a solution P which is a permutation matrix. Let

$$C = A^T * I - I * B^T = A * I - I * B, \ldots\ldots(3)$$

where $*$ denotes the Kronecker product and A, B are symmetric adjacency matrices of graphs

Thus, for a non-trivial permutation matrix solution to exist for equation (2), it is necessary that the matrix C is singular i.e. a necessary condition for the isomorphism of two graphs. Also, the N^2 x N^2 matrix, C holds all the information related to the isomorphism of graphs A, B.

Note: From the above Lemma 3, it is clear that the necessary and sufficient condition requires detailed characterization of null space of { 1, 0, -1 } matrix C. Based on the above necessary and sufficient condition for isomorphism of graphs, we prove the following interesting Lemma

Lemma 2: If a permutation matrix P exists such that equation (2) is satisfied i.e.

$$B\,P - P\,A \equiv \bar{0}$$, then
$$Rank(C) \leq N(N-1)$$

Proof: By the definition of the matrix C, it is clear that it is an N^2x N^2 matrix. By the Lemma 1 (i.e. Lemma 3 proved in [17]), it is clear that if the graphs are isomorphic, the matrix C is singular. Hence the determinant will be zero.

The purpose of this lemma is to bound the rank of the matrix. From the proof of Lemma (3) in [17], the system of equations specified by equation (2) can be converted into a system of linear equations of the form

$$C\,\bar{f} \equiv \bar{0} \quad where$$

$\bar{f}$ is Vec (P) i.e. vectorized form of a Permutation matrix P. By the definition of permutation matrix, $\bar{f}$ contains exactly N ones and all the other elements are zeroes. Hence, it readily follows that there are atleast N columns of the N^2x N^2 $matrix\ C$ that are linearly dependent. Thus, the rank of the matrix C can atmost be N^2- N QED

Note: The rank condition on C in the lemma is a stronger condition than singularity of the matrix C

Now, we design an algorithm to check the necessary condition on graph isomorphism provided by the Lemma 3 proved in [17].

(i) To determine the singularity of square matrix C, we can evaluate the determinant of such matrix. This can be done by an O(N^6) algorithm.

(ii) From computational linear algebra, efficient algorithms for determining the rank of a matrix are well known. The computational complexity of such algorithms is well worked out.

(iii) The algorithm to check the necessary and sufficient condition for isomorphism of graphs requires solving a constrained { 0, 1 } programming (integer programming) problem i.e. $\bar{f}$ is the vectorized version of the permutation matrix C.

- Isomorphism of Undirected Graphs: Algebraic Matrix Riccati equations:

From the known literature, a Non-symmetric Algebraic matrix Ricatti Equation (NARE) is of the form

XAX + XB + CX+ D$\equiv \bar{0}$, where the matrices

We consider the special case, where the unknown matrix X and { A, B,C,D } are compatible square matrices (i.e. matrix products are well defined).

Now, we consider the special case of equation (1), where the unknown matrix X is symmetric. Hence, we are interested in the equation of the following form:

X A X = D.

We call such a matrix equation as "structured algebraic matrix Ricatti equation". In [17], the author provided an approach to find an arbitrary solution of such a matrix equation. He also applied the solution method to find a permutation matrix solution in the case where the graphs are isomorphic based on a symmetric permutation matrix (and the matrices A,B being symmetric adjacency matrices).

Also, in the literature, a Sylvester matrix equation is of the following form:

$$X B + C X = F$$

where the matrices { X, B, C, F } are compatible matrices (i.e the matrix products are well defined). Large body of literature exists on solving such a matrix equation. In the case where F is a zero matrix, we call it a "structured Sylvester matrix equation".

We consider the special case of Sylvester matrix equation, where the unknown matrix is an arbitrary permutation matrix (not necessarily symmetric), B is an adjacency matrix of an undirected graph and C is the negative of adjacency matrix of an undirected graph and F is a zero matrix. As discussed in Lemma 1, two graphs are isomorphic if and only if a "structured Sylvester equation" is satisfied by an unknown matrix being a permutation matrix.

- NARE: Decomposition into Structured Sylvester Matrix Equation and Structured Riccati Equation:
  On closure observation, we realize that an arbitrary Non-Symmetric Algebraic Ricatti Equation (NARE) can be decomposed into "structured algebraic Ricatti equation" and "structured Sylvester matrix equation" (i.e. NARE can be expressed as sum of such structured matrix equations) . Explicitly, suppose, a matrix solution X exists such that

  $$X A X = D \qquad \text{and}$$
  $$XB + CX = 0$$

  are simultaneously satisfied. Then, we necessarily have that the matrix X satisfies NARE. For instance, in the case of isomorphic graphs with adjacency matrices A, D and the permutation matrix being symmetric, we have that

  $$X A X = D.$$

  Also, consider two subgraphs of such a graph that are isomorphic i.e. let B,-C be N x N symmetric adjacency matrices restricted to the subgraphs i.e. the edges from vertices in the subgrahs to vertices outside the subgraph are set to zero values. In this case, we have that

  $$XB = -CX \quad \text{or} \quad XB + CX = 0.$$

  Thus, with X being a permutation matrix (and A,B,C,D related to adjacency matrices), the Non Symmetric Algebraic Riccati equation is satisfied.

  Note: The hope is that if there is a unique solution to NARE based on permutation matrix, then isomorphism of graphs could be determined by such method.

  We now consider determination of isomorphism of STRUCTURED graphs based on the result specified in Lemma 1.
- Isomorphism of Regular Graphs: Structured Sylvester Matrix Equation: Diophantine Sylvester Matrix Equations:

Definition: An undirected graph is said to be "regular" if and only if the degree of all vertices is equal.

Thus, for a "regular" undirected graph, the row sums of the adjacency matrix are all equal.

In the case of isomorphic regular graphs, the following Lemma provides a necessary condition.

Lemma 3: Consider the adjacency matrices A, B of two "regular" isomorphic graphs. If a permutation matrix P exists such that equation (2) is satisfied i.e.

$$B\,P - P\,A \;\equiv \bar{0}$$, then

$$Rank(C) \leq N$$

Proof: Suppose the regular graphs associated with symmetric adjacency matrices A, B are isomorphic. Then, the degree of vertices in both the graphs are equal. Hence, letting $\bar{e}$ be a N x 1 vector (column) all of whose elements are equal to 1 , we have that

$$A\bar{e} = B\,\bar{e} = \theta\,\bar{e}\,.$$

We now claim that Q = $\bar{e}\,\bar{e}^T$ is a solution of the matrix equation in (1) i.e.

$$B\,Q - Q\,A \;\equiv \bar{0}\;\;.$$

We have that

$$B\,Q = B\,\bar{e}\,\bar{e}^T = \;\theta\,\bar{e}\,\bar{e}^T.$$

Also

$$Q\,A = \;\bar{e}\,\bar{e}^T\,A = \;\;\theta\,\bar{e}\,\bar{e}^T.$$

Hence the desired equation is satisfied by the all-ones matrix Q. Thus, as in the proof of Lemma 3 in [ ], we have that

$C\,\breve{e}$ = 0, where $\breve{e}$ *is a column vector of* N^2 *all ones.*

Also, as in Lemma 2,

$$C\,\bar{f} \equiv \bar{0} \quad where$$

$\bar{f}$ is Vec (P) i.e. vectorized form of a Permutation matrix P. It should be noted that $\bar{f}$ is a column vector with exactly N ones. Hence, we have that

$$C\,\breve{e}\; -\; C\,\bar{f} \equiv 0 = \;C\,\tilde{f},\,where$$

$\tilde{f}$ *is a column vector with exactly* $N(N-1)$ *non* $-$ *zero elements* (*being* $1, o, or\; -1$) . Hence the corresponding N(N-1) columns of matrix C are linearly dependent.

$$Thus\;\; Rank(C)\; \leq N \qquad\qquad\qquad\qquad QED.$$

Note: As noted earlier, the rank condition on C in the lemma is a stronger condition than singularity of the matrix C

Note: Using well known linear algebraic algorithms for the determination of rank of a matrix (of low computational complexity), it is possible to check if Rank(C) > N. In such case, the two regular graphs are not isomorphic.

Note: We realize that the column vectors { $\tilde{f}$, $\bar{f}$] both lie in the null space of the matrix C. Since, the null space of a matrix is a linear space, it follows that for arbitrary integers { α, β }, *we have that* $\alpha\tilde{f} + \beta\,\bar{f}$ lies in the null space of C. Such vectors contain the integer elements. The associated Sylvester matrix equations has the solution matrix with integer elements (i.e. a Diophantine Sylvester matrix equation).

- Sylvester Matrix Equation: Novel Results:

Consider a Sylvester matrix eqution i.e

$$G\,X + X\,H = J.$$

It is well known that a "unique" solution X exists if and only if { G, -H } donot share a common eigenvalue.
Consider the case where 'X' is an orthogonal matrix (e.g Permutation matrix)
i.e. $X^T = X^{-1}$. Thus, it readily follows that

$$X^T G\,X + H = X^T J \quad i.e.\ \ X^T G\,X - X^T J + H \ \equiv \bar{0}\,.$$

Such an equation has vague resemblance to the matrix Riccati equation. An orthogonal matrix solution to it exists if and only if the associated Sylvester matrix equation is satisfied. Also, study of arbitrary solutions (not necessarily orthogonal) to it could be of independent interest. Particularly, when { G, H, J } are adjacency matrices of undirected graphs, it could have graph-theoretic significance. Specifically, consider the Sylvester matrix equation

A X - X B = C, where

X is a permutation matrix and { A, B, C } are { 0, 1 } matrices. Particularly, the case where { A, B, C } are adjacency matrices seems to be interesting and can have graph theoretic significance. We can call it "PERTURBED GRAPH ISOMORPHISM PROBLEM".

Note: We can have interesting Diophantine Non-Symmetric Algebraic Matrix Riccati equations where the coefficient matrices are integer matrices. Such equations could find interesting applications like testing graph-isomophism.

- In the similar spirit of above discussion, consider an orthogonal matrix solution of NARE i.e

  A X + X B + X C X = D.

  It readily follows that

$$X^T A\,X + C\,X - X^T\,D + B \ \equiv \bar{0}.$$

  Such an equation (like NARE) could be interesting and can find some applications.

  Note: The results in this research paper can be applied to isomorphism of directed graphs.

4. CONCLUSIONS:

In this research paper, necessary condition for isomorphism of undirected graphs (stronger than those in [17]) based on "structured Sylvester equation" is derived. Associated polynomial time algorithm is discussed. Also, a strong necessary condition for isomorphism of regular graphs is discussed. It is shown that a "structured Algebraic Riccati equation" and "structured Sylvester matrix equation" are naturally related to the Non-Symmetric, Algebriac Riccati equation (NARE). Several interesting results related to Sylvester Matrix equation and NARE are discussed. Essentially graph isomorphism problems are related to linear and Non-linear matrix equations.

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